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# Solution of the Lorentz–Dirac equation based on a new momentum expression

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**Abstract.** The Lorentz–Dirac equation is solved based on a new momentum expression given by $p^\alpha = \frac{1}{c^2}(u_\mu p^\mu)u^\alpha + k\,\mathrm{d}u^\alpha/\mathrm{d}\tau$. This new momentum expression is the form proposed by Barut modified to satisfy the condition imposed by Dirac. The solution turns out to be well behaved without violating causality or causing runaway.

## 1. Introduction

The problem of the motion of a charged, radiating particle is notorious for its strange behaviour [1–3]. The solutions are either of a runaway type or of the type that will somehow violate causality. Thus, it seems to be believed that such a problem is essentially quantal and has no classical solutions without taking into account quantum effects which can yield sensible results. This may very well be the case. However, what exactly is to be called the quantum effect is rather debatable. Moreover, even if the final solution must be quantal, any valid classical theory that can produce a reasonable solution to this notorious problem shall be of some interest.

In the well known classic paper that established the Lorentz–Dirac equation, Dirac not only analysed the Lorentz–Dirac equation but also examined possible expressions of momentum that a radiating electron may take [1]. He concluded that a momentum expression $B_\mu$ may be any vector function of $u_\mu$ and its derivatives as long as it satisfies the condition

$$u^\mu \left(\frac{\mathrm{d}B_\mu}{\mathrm{d}\tau}\right) = 0 \tag{1.1}$$

where $u^\mu$ is the velocity four-vector. However, he decided to take the simplest possible form which is equivalent to the ordinary momentum expression $p^\mu = mu^\mu$. His justification was that other choices are more complicated than this simple one so that one would not expect them to apply to a simple thing like an electron. Whether an electron as a spinning and radiating particle is that simple is questionable.

In connection with a spinning particle there are theories that the momentum and the velocity may not be parallel [4, 5]. In regard to radiating electrons Barut proposed the following momentum velocity relation [6]

$$p_\mu = mu_\mu - (2e^2/3c^3)(\mathrm{d}u_\mu/\mathrm{d}\tau). \tag{1.2}$$

Although Barut proposed this new momentum expression in connection with the Lorentz–Dirac equation he did not solve the equation with this momentum expression. An examination of (1.2) shows that it is not compatible with the Dirac condition given by (1.1). If we take the essential idea of Barut's that the four-momentum for a radiating electron has a contribution from the four-acceleration, we may make a modification to satisfy the Dirac condition. We write

$$p^{\alpha} = \frac{1}{c^2}(u_{\nu} p^{\nu}) u^{\alpha} + \frac{\bar{h}}{kc^2}\frac{\mathrm{d}u^{\alpha}}{\mathrm{d}\tau}. \tag{1.3}$$

The $\bar{h}$ is introduced to make the unspecified constant $k$ dimensionless. We shall demonstrate that, using (1.3) as the definition of momentum for a radiating electron, the Lorentz–Dirac equation yields well behaved solutions without the defect of running away or violation of causality.

## 2. Solutions of the Lorentz–Dirac equation

The Lorentz–Dirac equation in force free space is given by [1–3]

$$\frac{\mathrm{d}p^{\alpha}}{\mathrm{d}\tau} = \frac{2e^2}{3c^3}\left[\frac{\mathrm{d}^2u^{\alpha}}{\mathrm{d}\tau^2} + \frac{1}{c^2}\left(\frac{\mathrm{d}u_{\nu}}{\mathrm{d}\tau}\right)\left(\frac{\mathrm{d}u^{\nu}}{\mathrm{d}\tau}\right)u^{\alpha}\right]. \tag{2.1}$$

For a better comparison with our new solution, we shall first briefly review how this old problem was treated by taking the usual

$$p^{\alpha} = mu^{\alpha}. \tag{2.2}$$

To solve equation (2.1) we assume a constant of the motion of (2.1) of the dimension of momentum to be given by

$$P^{\alpha} = \beta_1 u^{\alpha} + \beta_2 \frac{\mathrm{d}u^{\alpha}}{\mathrm{d}\tau} \tag{2.3}$$

where $\beta_1$ and $\beta_2$ are scalar functions of $\tau$ to be determined. From (2.1) and (2.2), we obtain

$$m\left(\frac{\mathrm{d}u_{\nu}}{\mathrm{d}\tau}\right)\left(\frac{\mathrm{d}u^{\nu}}{\mathrm{d}\tau}\right) = \frac{e^2}{3c^3}\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\left(\frac{\mathrm{d}u_{\nu}}{\mathrm{d}\tau}\right)\left(\frac{\mathrm{d}u^{\nu}}{\mathrm{d}\tau}\right)\right] \tag{2.4}$$

hence

$$\left(\frac{\mathrm{d}u_{\nu}}{\mathrm{d}\tau}\right)\left(\frac{\mathrm{d}u^{\nu}}{\mathrm{d}\tau}\right) = -\Gamma = -D^2 \exp\left(\frac{3mc^3\tau}{e^2}\right) \tag{2.5}$$

where $D$ is an integration constant. From (2.3), we obtain

$$(u_{\nu} P^{\nu}) = c^2 \beta_1 \tag{2.6}$$

$$\frac{\mathrm{d}\beta_1}{\mathrm{d}\tau} = -\frac{\beta_2}{c^2}\Gamma \tag{2.7}$$

$$P_{\nu} P^{\nu} = (Mc)^2 = c^2\beta_1^2 - \beta_2^2\Gamma \tag{2.8}$$

where $M$ is some constant of the dimension of mass. Using (2.5), we obtain from (2.8)

$$\beta_2^2 = \left(\frac{c}{D}\right)^2 \exp\left(\frac{-3mc^3\tau}{e^2}\right)(\beta_1^2 - M^2) \tag{2.9}$$

hence (2.7) becomes

$$\frac{\mathrm{d}\beta_1}{\mathrm{d}\tau} = \mp\left(\frac{D}{c}\right)\exp\left(\frac{3mc^3\tau}{2e^2}\right)(\beta_1^2 - M^2)^{1/2}. \tag{2.10}$$

Therefore, the results are given by

$$\beta_1 = M \cosh\left[\mp\frac{2De^2}{3mc^4}\exp\left(\frac{3mc^3}{2e^2}\tau\right)\right] \tag{2.11}$$

$$\beta_2 = \pm M\left(\frac{c}{D}\right)\exp\left(\frac{-3mc^3}{2e^2}\tau\right)\sinh\left[\mp\frac{2De^2}{3mc^4}\exp\left(\frac{3mc^3}{2e^2}\tau\right)\right] \tag{2.12}$$

and

$$u^\alpha = Q^\alpha \sinh\left[\frac{\mp 2De^2}{3mc^4}\exp\left(\frac{3mc^3}{2e^2}\tau\right)\right] + \frac{P^\alpha}{M}\cosh\left[\mp\frac{2De^2}{3mc^4}\exp\left(\frac{3mc^3}{2e^2}\tau\right)\right] \tag{2.13}$$

where $Q^\alpha$ are integration constants satisfying the following relations

$$Q_\nu Q^\nu = -c^2 \tag{2.14}$$

$$Q_\nu P^\nu = 0. \tag{2.15}$$

From (2.13) it is obvious that as $\tau \to \infty$, $u^\alpha \to \infty$. This is the notorious runaway solution. Were the sign in front of the right-hand side of (2.1) a minus, rather than a positive sign as it is, the final result would be

$$u^\alpha = Q^\alpha \sinh\left[\pm\frac{2De^2}{3mc^4}\exp\left(\frac{-3mc^3\tau}{2e^2}\right)\right] + \frac{P^\alpha}{M}\cosh\left[\pm\frac{2De^2}{3mc^4}\exp\left(\frac{-3mc^3\tau}{2e^2}\right)\right]. \tag{2.16}$$

Result (2.16) appears to be well behaved as $\tau \to \infty$. However, Dirac concluded that it is not possible to tamper with the sign in any relativistic way without getting into further trouble.

Without tampering with the sign, there is still one option to modify the Lorentz–Dirac equation which Dirac himself already noted, namely to consider other possible momentum expressions. We shall now try with the expression (1.3) as follows.

Substituting (1.3) into (2.1), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}p^\alpha &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{1}{c^2}(u_\nu p^\nu)\right]u^\alpha + \left[\frac{1}{c^2}(u_\nu p^\nu)\right]\frac{\mathrm{d}u^\alpha}{\mathrm{d}\tau} + \frac{\hbar}{kc^2}\frac{\mathrm{d}^2u^\alpha}{\mathrm{d}\tau^2}\\ &= \frac{2e^2}{3c^3}\left[\frac{\mathrm{d}^2u^\alpha}{\mathrm{d}\tau^2} + \frac{1}{c^2}\left(\frac{\mathrm{d}u_\nu}{\mathrm{d}\tau}\right)\left(\frac{\mathrm{d}u^\nu}{\mathrm{d}\tau}\right)u^\alpha\right].\end{aligned} \tag{2.17}$$

Contracting (2.17) with $u_\alpha$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(u_\nu p^\nu) + \frac{\hbar}{kc^2}u_\nu\frac{\mathrm{d}^2u^\nu}{\mathrm{d}\tau^2} = 0. \tag{2.18}$$

Thus, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(u_\nu p^\nu) = \frac{\hbar}{kc^2}\left(\frac{\mathrm{d}u_\nu}{\mathrm{d}\tau}\right)\left(\frac{\mathrm{d}u^\nu}{\mathrm{d}\tau}\right) = -\frac{\hbar}{kc^2}\Gamma. \tag{2.19}$$

Equation (2.17) can now be rewritten as

$$\frac{\mathrm{d}u^\alpha}{\mathrm{d}\tau} = c^2(u_\nu p^\nu)^{-1}\left(\frac{2e^2}{3c^3} - \frac{\hbar}{kc^2}\right)\left[\frac{\mathrm{d}^2u^\alpha}{\mathrm{d}\tau^2} + \frac{1}{c^2}\left(\frac{\mathrm{d}u_\nu}{\mathrm{d}\tau}\right)\left(\frac{\mathrm{d}u^\nu}{\mathrm{d}\tau}\right)u^\alpha\right]. \tag{2.20}$$

Contracting (2.20) with $\frac{\mathrm{d}u_\alpha}{\mathrm{d}\tau}$ we obtain

$$(-\Gamma) = \frac{c^2}{2}(u_\nu p^\nu)^{-1}\left(\frac{2e^2}{3c^3} - \frac{\hbar}{kc^2}\right)\frac{\mathrm{d}}{\mathrm{d}\tau}(-\Gamma). \tag{2.21}$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(u_\nu p^\nu) = -\frac{\hbar}{kc^2}\Gamma = \frac{c^2}{2}\left(\frac{2e^2}{3c^3} - \frac{\hbar}{kc^2}\right)\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\ln\Gamma. \tag{2.22}$$

The solution to (2.22) is given by

$$\Gamma = (\lambda\sigma c)^2(W^2 - 1) \tag{2.23}$$

$$W = \frac{\exp(\lambda\tau) + \xi\exp(-\lambda\tau)}{\exp(\lambda\tau) - \xi\exp(-\lambda\tau)} \tag{2.24}$$

$$\sigma = \left[1 - \frac{2ke^2}{3\hbar c}\right]^{1/2} \tag{2.25}$$

where $\lambda$ and $\xi$ are integration constants to be determined from initial conditions. Integrating (2.23) or using (2.21) we obtain

$$(u_\nu p^\nu) = c^2\lambda\left(\frac{\hbar}{kc^2} - \frac{2e^2}{3c^3}\right)W. \tag{2.26}$$

When (2.23)–(2.25) are substituted into (2.20) the apparently nonlinear equation becomes a linear equation of second order in $u^\alpha$ with time-dependent functions as the coefficients. To solve (2.20) we make the assumption that

$$P^\alpha = A_1 u^\alpha + A_2\frac{\mathrm{d}u^\alpha}{\mathrm{d}\tau} \tag{2.27}$$

is a constant of the motion with $A_1$ and $A_2$ some scalar functions of $\tau$ to be determined. We then have

$$P_\nu P^\nu = (Mc)^2 = c^2A_1^2 - A_2^2\Gamma \tag{2.28}$$

$$c^2A_1 = u_\nu P^\nu \tag{2.29}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(u_\nu P^\nu) = c^2\frac{\mathrm{d}A_1}{\mathrm{d}\tau} = -A_2\Gamma. \tag{2.30}$$

Thus

$$A_2^2 = c^2(A_1^2 - M^2)\Gamma^{-1} \tag{2.31}$$

$$\frac{\mathrm{d}A_1}{\mathrm{d}\tau} = -\frac{A_2}{c^2}\,\Gamma = -\lambda\sigma[(A_1^2 - M^2)(W^2-1)]^{1/2}. \tag{2.32}$$

The solution to (2.32) is given by

$$A_1 = M\cosh[\sigma\cosh^{-1}(W)]. \tag{2.33}$$

Thus from (2.31), we have

$$A_2 = \frac{M}{\lambda\sigma}[(W^2-1)]^{-1/2}\sinh[\sigma\cosh^{-1}(W)]. \tag{2.34}$$

With $A_1$ and $A_2$ given by (2.33) and (2.34), equation (2.27) can now be solved. The solution to (2.27) is given by

$$u^\alpha = (P^\alpha/M)\cosh[\sigma\cosh^{-1}(W)] + \omega^\alpha\sinh[\sigma\cosh^{-1}(W)] \tag{2.35}$$

where the $\omega^\alpha$ are integration constants satisfying the following relations

$$\omega_\nu\omega^\nu = -c^2 \tag{2.36}$$

$$\omega_\nu P^\nu = 0. \tag{2.37}$$

Differentiating (2.35) we have

$$\frac{\mathrm{d}u^\alpha}{\mathrm{d}\tau} = -\lambda\sigma[W^2-1]^{1/2}\{(P^\alpha/M)\sinh[\sigma\cosh^{-1}(W)] + \omega^\alpha\cosh[\sigma\cosh^{-1}(W)]\}. \tag{2.38}$$

Using (2.26) we can write (1.3) as

$$p^\alpha = \frac{\hbar}{kc^2}\left[(\lambda\sigma^2 W)u^\alpha + \frac{\mathrm{d}u^\alpha}{\mathrm{d}\tau}\right]. \tag{2.39}$$

Thus, equation (2.17) is completely solved except for the determination of integration constants in terms of the initial conditions. The solutions can be seen to be well behaved.

To determine the integration constants $\xi$ and $\lambda$ in terms of the initial conditions, we note that from (1.3), (2.26) and (2.23) that the following relations can be established

$$p_\nu p^\nu + \left(\frac{\hbar}{kc^2}\right)^2 \Gamma = \frac{1}{c^2}(u_\nu p^\nu)^2 = c^2\lambda^2\sigma^4 W^2 \left(\frac{\hbar}{kc^2}\right)^2 = \left(\frac{\hbar}{kc^2}\right)^2 \sigma^2 W^2 \Gamma (W^2-1)^{-1}. \tag{2.40}$$

Thus

$$W^2 = \left[p_\nu p^\nu + \left(\frac{\hbar}{kc^2}\right)^2 \Gamma\right]\left[p_\nu p^\nu + \left(\frac{\hbar}{kc^2}\right)^2 \Gamma(1-\sigma^2)\right]^{-1}. \tag{2.41}$$

Now setting $\tau = 0$ (2.41) becomes

$$W_0^2 = [(1+\xi)/(1-\xi)]^2 = \left\{\left[p_\nu p^\nu + \left(\frac{\hbar}{kc^2}\right)^2 \Gamma\right]\left[p_\nu p^\nu + \left(\frac{\hbar}{kc^2}\right)^2 \Gamma(1-\sigma^2)\right]^{-1}\right\}_0 . \tag{2.42}$$

Hence

$$\xi = (W_0 - 1)/(W_0 + 1) \tag{2.43}$$

and from (2.23)

$$\lambda = (c\sigma)^{-1}[\Gamma_0/(W_0^2 - 1)]^{1/2}. \tag{2.44}$$

To determine $P^\alpha$ and $\omega^\alpha$ in terms of the initial conditions we set $\tau = 0$ in (2.35) and (2.38) to obtain

$$u^\alpha(0) = (P^\alpha/M)\cosh[\sigma\cosh^{-1}(W_0)] + \omega^\alpha \sinh[\sigma\cosh^{-1}(W_0)] \tag{2.45}$$

$$\frac{\mathrm{d}u^\alpha(0)}{\mathrm{d}\tau} = -\lambda\sigma[W_0^2-1]^{1/2}\{(P^\alpha/M)\sinh[\sigma\cosh^{-1}(W_0)] + \omega^\alpha\cosh[\sigma\cosh^{-1}(W_0)]\}. \tag{2.46}$$

Then we obtain

$$\frac{P^\alpha}{M} = u^\alpha(0)\cosh[\sigma\cosh^{-1}(W_0)] + \frac{\mathrm{d}u^\alpha(0)}{\mathrm{d}\tau}(\lambda\sigma)^{-1}[W_0^2-1]^{-1/2}\sinh[\sigma\cosh^{-1}(W_0)] \tag{2.47}$$

$$\omega^\alpha = -u^\alpha(0)\sinh[\sigma\cosh^{-1}(W_0)] - \frac{\mathrm{d}u^\alpha(0)}{\mathrm{d}\tau}(\lambda\sigma)^{-1}[W_0^2-1]^{-1/2}\cosh[\sigma\cosh^{-1}(W_0)]. \tag{2.48}$$

Now setting $\tau \to \infty$ we have asymptotically

$$u^\alpha(\infty) = \frac{P^\alpha}{M} \qquad \frac{\mathrm{d}u^\alpha(\infty)}{\mathrm{d}\tau} = 0 \qquad W(\infty) = 1 \qquad p^\alpha(\infty) = \left(\frac{\hbar}{kc^2}\right)\lambda\sigma^2 u^\alpha(\infty). \tag{2.49}$$

If we identify the $M$ as the asymptotic mass of the charged particle and also $P^\alpha = p^\alpha(\infty)$ then from (2.49) we obtain

$$M = \left(\frac{\hbar}{kc^2}\right)\lambda\sigma^2. \tag{2.50}$$

Therefore if $k$ is known all the integration constants are given in terms of the initial conditions. The constant $k$ is introduced from the beginning in (1.3) as something intrinsic and is not one of the integration constants. Thus it should not be expected to be fixed by the initial conditions. For our purpose to demonstrate that the solutions of the Lorentz–Dirac equation using (1.3) are well behaved, the precise value of $k$ does not matter much. However, for the expression (1.3) to acquire fundamental significance some way must be found to definitely fix the value of $k$. We do not have a completely satisfactory way for such a determination and shall leave this problem open.

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